MARGINAL RELEVANCE OF DISORDER FOR PINNING MODELS

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Abstract. The effect of disorder on pinning and wetting models has attracted much attention in theoretical physics (*e.g.* [16, 11]). In particular, it has been predicted on the basis of the *Harris criterion* that disorder is *relevant* (annealed and quenched model have different critical points and critical exponents) if the return probability exponent α, a positive number that characterizes the model, is larger than $1/2$. Weak disorder has been predicted to be *irrelevant* (*i.e.* coinciding critical points and exponents) if $\alpha < 1/2$. Recent mathematical work (in particular [2, 10, 21, 22]) has put these predictions on firm grounds. In *renormalization group* terms, the case $\alpha = 1/2$ is a *marginal* case and there is no agreement in the literature as to whether one should expect disorder relevance [11] or irrelevance [16] at marginality. The question is particularly intriguing also because the case $\alpha = 1/2$ includes the classical models of two-dimensional wetting of a rough substrate, of pinning of directed polymers on a defect line in dimension $(3+1)$ or $(1+1)$ and of pinning of an heteropolymer by a point potential in three-dimensional space. Here we prove disorder relevance both for the general $\alpha = 1/2$ pinning model and for the hierarchical version of the model proposed in [11], in the sense that we prove a shift of the quenched critical point with respect to the annealed one. In both cases we work with Gaussian disorder and we show that the shift is at least of order $\exp(-1/\beta^4)$ for β small, if β^2 is the disorder variance.

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1. Introduction

1.1. **Wetting and pinning on a defect line in $(1+1)$-dimensions.** The intense activity aiming at understanding phenomena like wetting in two dimensions [1] and pinning of polymers by a defect line [15] has led several people to focus on a class of simplified models based on random walks. In order to describe more realistic, spatially inhomogeneous situations, these models include disordered interactions. While a very substantial amount of work has been done, it is quite remarkable that some crucial issues are not only mathematically open (which is not surprising given the presence of disorder), but also controversial in the physics literature.

Let us start by introducing the most basic, and most studied, model in the class we consider (it is the case considered in [16, 11], but also in [6, 17, 23, 24, 29, 30], up to some inessential details, although the notations used by the various authors are quite different). Let $S = \{S_0, S_1, \ldots\}$ be a simple symmetric random walk on $\mathbb{Z}$, *i.e.*, $S_0 = 0$ and $\{S_n - S_{n-1}\}_{n\in\mathbb{N}}$ is an IID sequence (with law $\mathbf{P}$) of random variables taking values ± 1 with probability $1/2$. It is better to take a directed walk viewpoint, that is to consider the process $\{(n, S_n)\}_{n=0,1,\ldots}$. This random walk is the *free model* and we want to understand

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the situation where the walk interacts with a substrate or with a defect line that provides *disordered* (*e.g.* random) rewards/penalties each time the walk hits it (see Fig. 1). The walk may or may not be allowed to take negative values: we call *pinning on a defect line* the first case and *wetting of a substrate* the second one. It is by now well understood that these two cases are equivalent and we briefly discuss the wetting case only in the caption of Figure 1: the general model we will consider covers both wetting and pinning cases. The interaction is introduced via the Hamiltonian

$$H_{N,\omega}(S) := -\sum_{n=1}^{N} \left(\beta\omega_n + h - \log \mathbb{E}(\exp(\beta\omega_1))\right) \mathbf{1}_{\{S_n=0\}}, \tag{1.1}$$

where $N \in 2\mathbb{N}$ is the system size, h (homogeneous pinning potential) is a real number, $\omega := \{\omega_1, \omega_2, \ldots\}$ is a sequence of IID centered random variables with finite exponential moments (in this work, we will restrict to the Gaussian case), $\beta \geq 0$ is the disorder strength and $\mathbb{E}$ denotes the average with respect to ω. It will be soon clear what is the notational convenience in introducing the non-random term $\log \mathbb{E}(\exp(\beta\omega_1))$ (which could be absorbed into h anyway).

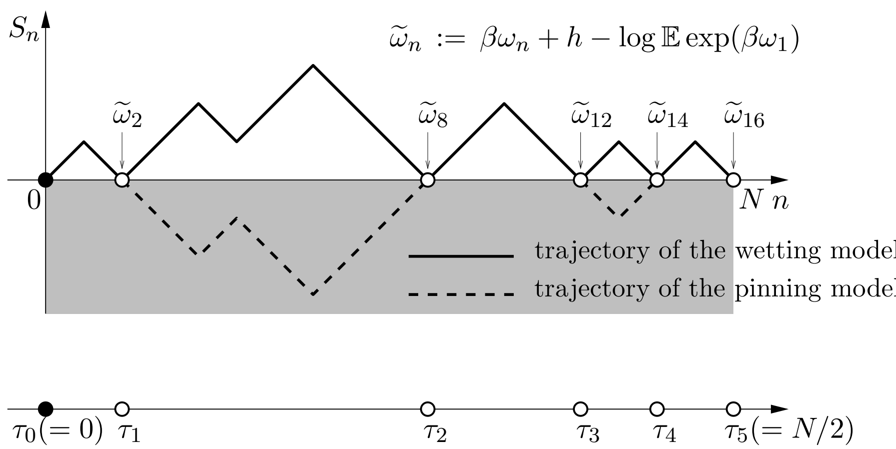

FIGURE 1. In the top a random walk trajectory, pinned at N, which is not allowed to enter the lower half-plane (the shadowed region should be regarded as a wall). The trajectory collects the *charges* $\widetilde{\omega}_n$ when it hits the wall. The question is whether the rewards/penalties collected pin the walk to the wall or not. The precise definition of the wetting model is obtained by multiplying the numerator in the right-hand side of (1.2) by the indicator function of the event $\{S_j \geq 0,\, j = 1, \ldots, N\}$ (and consequently modifying the partition function $Z_{N,\omega}$). This model is actually equivalent to the model (1.2) without a wall, whose trajectories (dashed line) can visit the lower half plane, provided that h is replaced by $h - \log 2$ (see [18, Ch. 1]). The bottom part of the figure illustrates the simple but crucial point that the energy of the model depends only on the location of the points of contact between walk and wall (or defect line); such points form a renewal process, giving thus a natural generalized framework in which to tackle the problem. In order to circumvent the annoying periodicity two of the simple random walk we set $\tau_0 = 0$ and $\tau_{j+1} := \inf\{n/2 \geq \tau_j : S_n = 0\}$. From the renewal process standpoint, introducing a wall just leads to a *terminating* renewal (see text).

The Gibbs measure $\mathbf{P}_{N,\omega}$ for the pinning model is then defined as

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{e^{-H_{N,\omega}(S)}\mathbf{1}_{\{S_N=0\}}}{Z_{N,\omega}} \tag{1.2}$$

and of course $Z_{N,\omega} := \mathbf{E}[\exp(-H_{N,\omega}(S))\mathbf{1}_{\{S_N=0\}}]$, where $\mathbf{E}$ denotes expectation with respect to the simple random walk measure $\mathbf{P}$. Note that we imposed the boundary condition $S_N = 0 = S_0$ (just to be consistent with the rest of the paper). It is well known that the model undergoes a localization/delocalization transition as h varies: if h is larger than a certain threshold value $h_c(\beta)$ (*quenched critical point*) then, under the Gibbs measure, the system is *localized*: the contact fraction, defined as

$$\frac{1}{N}\mathbf{E}_{N,\omega}\left[\sum_{n=1}^{N}\mathbf{1}_{\{S_n=0\}}\right], \tag{1.3}$$

tends to a positive limit for $N \to \infty$. On the other hand, for $h < h_c(\beta)$ the system is *delocalized*, *i.e.*, the limit is zero.

The result we just stated is true also in absence of disorder ($\beta = 0$) and a remarkable fact for the homogeneous (*i.e.* non-disordered) model is that it is exactly solvable ([14, 18] and references therein). In particular, we know that $h_c(0) = 0$, *i.e.*, an arbitrarily small reward is necessary and sufficient for pinning, and that the free energy behaves quadratically close to criticality. If now we consider the *annealed measure* corresponding to (1.2), that is the model in which one replaces both $\exp(-H_{N,\omega}(S))$ and $Z_{N,\omega}$ by their averages with respect to ω, one readily realizes that the annealed model is a homogeneous model, and precisely the one we obtain by setting $\beta = 0$ in (1.2). Therefore one finds that the *annealed critical point* $h_c^a(\beta)$ equals 0 for every β, and that the *annealed free energy* $\textsc{f}^a(\beta, h)$ behaves, for $h \searrow 0$, like $\textsc{f}^a(\beta, h) \sim const \times h^2$, while it is zero for $h \leq 0$.

Very natural questions are: does $h_c(\beta)$ differ from $h_c^a(\beta)$? Are quenched and annealed critical exponents different? As we are going to explain, the first question finds contradictory answers in the literature, while no clear-cut statement can really be found about the second. Below we are going to argue that these two questions are intimately related, but first we make a short detour in order to define a more general class of models. It is in this more general context that the role of the disorder and the specificity of the simple random walk case can be best appreciated.

1.2. **Reduction to renewal-based models.** As argued in the caption of Figure 1, the basic underlying process is the *point process* $\tau := \{\tau_0, \tau_1, \ldots\}$, which is a renewal process (that is $\{\tau_n - \tau_{n-1}\}_{n\in\mathbb{N}}$ is an IID sequence of integer-valued random variables). We set $K(n) := \mathbf{P}(\tau_1 = n)$. It is well known that, for the simple random walk case, $\sum_{n\in\mathbb{N}} K(n) = 1$ (the walk is recurrent) and $K(n) \overset{n\to\infty}{\sim} 1/(\sqrt{4\pi}n^{3/2})$. This suggests the natural generalized framework of models based on discrete renewal processes such that

$$\sum_{n\in\mathbb{N}} K(n) \leq 1 \quad \text{and} \quad K(n) \overset{n\to\infty}{\sim} \frac{C_K}{n^{1+\alpha}}, \tag{1.4}$$

with $C_K > 0$ and $\alpha > 0$. We are of course employing the standard notation $a_n \sim b_n$ for $\lim a_n/b_n = 1$. The case $\sum_{n\in\mathbb{N}} K(n) < 1$ refers to transient (or *terminating*) renewals (of which the wetting case is an example), see also Remark 3.2 below. This framework includes for example the simple random walk in $d \geq 3$, for which $\sum_{n\in\mathbb{N}} K(n) < 1$ and $\alpha = (d/2) - 1$, but it is of course much more general. We will come back with more details on this model, but let us just say now that the definition of the Gibbs measure is given in this case by (1.1)-(1.2), with S replaced by τ in the left-hand side and with the event $\{S_n = 0\}$ replaced by the event $\{\text{there is } j \text{ such that } \tau_j = n\}$.

1.3. **Harris criterion and disorder relevance: the state of the art.** The questions mentioned at the end of Section 1.1 are typical questions of disorder relevance, *i.e.*, of stability of critical properties with respect to (weak) disorder. In renormalization group language, one is asking whether or not disorder drives the system towards a new fixed point. A heuristic tool which was devised to give an answer to such questions is the *Harris criterion* [25], originally proposed for random ferromagnetic Ising models. The Harris criterion states that disorder is relevant if the specific heat exponent of the pure system is positive, and irrelevant if it is negative. In case such critical exponent is zero (this is called a *marginal case*), the Harris criterion gives no prediction and a case-by-case delicate analysis is needed. Now, it turns out that the random pinning model described above is a marginal case, and from this point of view it is not surprising that the question of disorder relevance is not solved yet, even on heuristic grounds: in particular, the authors of [16] (and then also [23, 24] and, very recently, [17]) claimed that for small β the quenched critical point coincides with the annealed one (with our conventions, this means that both are zero), while in [11] it was concluded that they differ for every $\beta > 0$, and that their difference is of order $\exp(-const/\beta^2)$ for β small (we mention [6, 29, 30] which support this second possibility). Note that such a quantity is smaller than any power of β, and therefore vanishes at all orders in weak-disorder perturbation theory (this is also typical of marginal cases).

In an effort to reduce the problem to its core, beyond the difficulties connected to the random walk or renewal structure, a *hierarchical pinning model*, defined on a diamond lattice, was introduced in [11]. In this case, the laws of the partition functions for the systems of size N and $2N$ are linked by a simple recursion. The role of α is played here by a real parameter $B \in (1, 2)$, which is related to the geometry of the hierarchical lattice. Also in this case, the Harris criterion predicts that disorder is relevant in a certain regime (here, $B < B_c := \sqrt{2}$) and irrelevant in another ($B > B_c$), while $B = B_c$ is the marginal case where the specific heat critical exponent of the pure model vanishes. Again, the authors of [11] predicted that disorder is marginally relevant for $B = B_c$, and that the difference between annealed and quenched critical point behaves like $\exp(-const/\beta^2)$ for β small (they gave also numerical evidence that the critical exponent is modified by disorder).

Let us mention that hierarchical models based on diamond lattices have played an important role in elucidating the effect of disorder on various statistical mechanics models: we mention for instance [9].

The mathematical comprehension of the question of disorder relevance in pinning models has witnessed remarkable progress lately. First of all, it was proven in [21] that an arbitrarily weak (but extensive) disorder changes the critical exponent if $\alpha > 1/2$ (the analogous result for the hierarchical model was proven in [28]). Results concerning the critical points came later: in [2, 31] it was proven that if $\alpha < 1/2$ then $h_c(\beta) = 0$ (and the quenched critical exponent coincides with the annealed one) for β sufficiently small (the analogous result for the hierarchical model was given in [19]). Finally, the fact that $h_c(\beta) > 0$ for every $\beta > 0$ (together with the correct small-β behavior) in the regime where the Harris criterion predicts disorder relevance was proven in [19] in the hierarchical set-up, and then in [10, 3] in the non-hierarchical one. One can therefore safely say that the comprehension of the relevance question is by now rather solid, *except in the marginal case* (of course some problems remain open, for instance the determination of the value of the quenched critical exponent in the relevant disorder regime, beyond the bounds proved in [21]).

1.4. **Marginal relevance of disorder.** In this work, we solve the question of disorder relevance for the marginal case $\alpha = 1/2$ (or $B = B_c$ in the hierarchical situation), showing that *quenched and annealed critical points differ for every disorder strength* $\beta > 0$. We also give a quantitative bound, $h_c(\beta) \geq \exp(-const/\beta^4)$ for β small, which is however presumably not optimal. The method we use is a non-trivial extension of the *fractional moment – change of measure method* which already allowed to prove disorder relevance for $B < B_c$ in [19] or for $\alpha > 1/2$ in [10]. A few words about the evolution of this method may be useful to the reader. The idea of estimating non-integer moments of the partition function of disordered systems is not new: consider for instance [7] in the context of directed polymers in random environment, or [5] in the context of Anderson localization (in the latter case one deals with non-integer moments of the propagator). However, the power of non-integer moments in pinning/wetting models was not appreciated until [32], where it was employed to prove, among other facts, that quenched and annealed critical points differ for large β, irrespective of the value of $\alpha \in (0, \infty)$. The new idea which was needed to treat the case of weak disorder (small β) was instead introduced in [19, 10], and it is a change-of-measure idea, coupled with an *iteration procedure*: one changes the law of the disorder ω in such a way that the new and the old laws are very close in a certain sense, but under the new one it is easier to prove that the fractional moments of the partition function are small. In the relevant disorder regime, $\alpha > 1/2$ or $B < B_c$, it turns out that it is possible to choose the new law so that the ω_n's are still IID random variables, whose law is simply tilted with respect to the original one. This tilting procedure is bound to fail if applied for arbitrarily large volumes, but having such bounds for sufficiently large, but finite, system sizes is actually sufficient because of an iteration argument (which appears very cleanly in the hierarchical set-up).

In order to deal with the marginal case we will instead introduce a long-range anti-correlation structure for the ω-variables. Such correlations are carefully chosen in order to reflect the structure of the two-point function of the annealed model and, in the non-hierarchical case, they are restricted, via a coarse-graining procedure inspired by [33], only to suitable *disorder pockets*.

We mention also that one of us [27] proved recently that disorder is marginally relevant in a different version of the hierarchical pinning model. What simplifies the task in that case is that the Green function of the model is spatially inhomogeneous and one can take advantage of that by tilting the ω-distributions in a inhomogeneous way (keeping the ω's independent). The Green function of the hierarchical model proposed in [11] is instead constant throughout the system and inhomogeneous tilting does not seem to be of help (as it does not seem to be of help in the non-hierarchical case, since it does not match with the coarse graining procedure).

The paper is organized as follows: the hierarchical (resp. non-hierarchical) pinning model is precisely defined in Section 2 (resp. in Section 3), where we also state our result concerning marginal relevance of disorder. Such result is proven in Section 4 in the hierarchical case, and in Section 5 in the non-hierarchical one.

In order not to hide the novelty of the idea with technicalities, we restrict ourselves to Gaussian disorder and, in the case of the non-hierarchical model, we do not treat the natural generalization where $K(\cdot)$ is of the form $K(n) = L(n)/n^{3/2}$ with $L(\cdot)$ a slowly varying function [13, VIII.8]. We plan to come back to both issues in a forthcoming paper [20].

2. The hierarchical model

Let $1 < B < 2$. We study the following iteration which transforms a vector $\{R_n^{(i)}\}_{i\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}}$ into a new vector $\{R_{n+1}^{(i)}\}_{i\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}}$:

$$R_{n+1}^{(i)} = \frac{R_n^{(2i-1)} R_n^{(2i)} + (B-1)}{B}, \tag{2.1}$$

for $n \in \mathbb{N} \cup \{0\}$ and $i \in \mathbb{N}$.

In particular, we are interested in the case in which the initial condition is random and given by $R_0^{(i)} = e^{\beta\omega_i - \beta^2/2 + h}$, with $\omega := \{\omega_i\}_{i\in\mathbb{N}}$ a sequence of IID standard Gaussian random variables and $h \in \mathbb{R}, \beta \geq 0$. We denote by $\mathbb{P}$ the law of ω and by $\mathbb{E}$ the corresponding average. In this case, it is immediate to realize that for every given n the random variables $\{R_n^{(i)}\}_{i\in\mathbb{N}}$ are IID. We will study the behavior for large n of $X_n := R_n^{(1)}$.

It is easy to see that the average of X_n satisfies the iteration

$$\mathbb{E}(X_{n+1}) = \frac{(\mathbb{E}X_n)^2 + (B-1)}{B}, \tag{2.2}$$

with initial condition $\mathbb{E}(X_0) = e^h$. The map (2.2) has two fixed points: a stable one, $\mathbb{E}X_n = (B-1)$, and an unstable one, $\mathbb{E}X_n = 1$. This means that if $0 \leq \mathbb{E}X_0 < 1$ then $\mathbb{E}X_n$ tends to $(B-1)$ when $n \to \infty$, while if $\mathbb{E}X_0 > 1$ then $\mathbb{E}X_n$ tends to $+\infty$.

Remark 2.1. In [11] and [19], the model with $B > 2$ was considered. However, the cases $B \in (1,2)$ and $B \in (2,\infty)$ are equivalent. Indeed, if $R_n^{(i)}$ satisfies (2.1) with $B > 2$, it is immediate to see that $\widehat{R}_n^{(i)} := R_n^{(i)}/(B-1)$ satisfies the same iteration but with B replaced by $\widehat{B} := B/(B-1) \in (1,2)$. In this work, we prefer to work with $B \in (1,2)$ because things turn out to be notationally simpler (*e.g.*, the annealed critical point (defined in the next section) turns out to be 0 rather than $\log(B-1)$). In the following, whenever we refer to results from [19] we give them for $B \in (1,2)$.

2.1. **Quenched and annealed free energy and critical point.**

The random variable X_n is interpreted as the partition function of the hierarchical random pinning model on a diamond lattice of generation n (we refer to [11] for a clear discussion of this connection). The *quenched free energy* is then defined as

$$\textsc{f}(\beta,h) := \lim_{n\to\infty} \frac{1}{2^n} \mathbb{E}\log X_n. \tag{2.3}$$

In [19, Th. 1.1] it was proven, among other facts, that for every $\beta \geq 0, h \in \mathbb{R}$ the limit (2.3) exists and it is non-negative. Moreover, $\textsc{f}(\beta,\cdot)$ is convex and non-decreasing. On the other hand, the *annealed free energy* is by definition

$$\textsc{f}^a(\beta,h) := \lim_{n\to\infty} \frac{1}{2^n} \log \mathbb{E}X_n. \tag{2.4}$$

Since the initial condition of (2.1) was normalized so that $\mathbb{E}X_0 = e^h$, it is easy to see that the annealed free energy is nothing but the free energy of the non-disordered model:

$$\textsc{f}^a(\beta,h) = \textsc{f}(0,h). \tag{2.5}$$

Non-negativity of the free energy allows to define the *quenched critical* point in a natural way, as

$$h_c(\beta) := \inf\{h \in \mathbb{R} : \textsc{f}(\beta,h) > 0\}, \tag{2.6}$$

and analogously one defines the *annealed critical point* $h_c^a(\beta)$. In view of observation (2.5), one sees that $h_c^a(\beta) = h_c(0)$. Monotonicity and convexity of $\mathrm{F}(\beta, \cdot)$ imply that $\mathrm{F}(\beta, h) = 0$ for $h \le h_c(\beta)$.

For the annealed system, the critical point and the critical behavior of the free energy around it are known (see [11] or [19, Th. 1.2]). What one finds is that for every $B \in (1, 2)$ one has $h_c(0) = 0$, and there exists $c := c(B) > 0$ such that for all $0 \le h \le 1$

$$c(B)^{-1} h^{1/\alpha} \le \mathrm{F}(0, h) \le c(B) h^{1/\alpha}, \tag{2.7}$$

where

$$\alpha := \frac{\log(2/B)}{\log 2} \in (0, 1). \tag{2.8}$$

Observe that α is decreasing as a function of B, and equals $1/2$ for $B = B_c := \sqrt{2}$.

2.2. **Disorder relevance or irrelevance.** The main question we are interested in is whether quenched and annealed critical points differ, and if yes how does their difference behave for small disorder. Jensen's inequality, $\mathbb{E} \log X_n \le \log \mathbb{E} X_n$, implies in particular that $\mathrm{F}(\beta, h) \le \mathrm{F}(0, h)$ so that $h_c(\beta) \ge h_c(0) = 0$. Is this inequality strict?

In [19] a quite complete picture was given, except in the marginal case $B = B_c$ which was left open:

Theorem 2.2. [19, Th. 1.4] *If $1 < B < B_c$, $h_c(\beta) > 0$ for every $\beta > 0$ and there exists $c_1 > 0$ such that for $0 \le \beta \le 1$*

$$c_1 \beta^{2\alpha/(2\alpha-1)} \le h_c(\beta) \le c_1^{-1} \beta^{2\alpha/(2\alpha-1)}. \tag{2.9}$$

If $B = B_c$ there exists $c_2 > 0$ such that for $0 \le \beta \le 1$

$$h_c(\beta) \le \exp(-c_2/\beta^2). \tag{2.10}$$

If $B_c < B < 2$ there exists $\beta_0 > 0$ such that $h_c(\beta) = 0$ for every $0 < \beta \le \beta_0$.

The main result of the present work is that in the marginal case, the two critical points *do differ* for every disorder strength:

Theorem 2.3. *Let $B = B_c$. For every $0 < \beta_0 < \infty$ there exists a constant $0 < c_3 := c_3(\beta_0) < \infty$ such that for every $0 < \beta \le \beta_0$*

$$h_c(\beta) \ge \exp(-c_3/\beta^4). \tag{2.11}$$

3. The non-hierarchical model

We let $\tau := \{\tau_0, \tau_1, \ldots\}$ be a renewal process of law $\mathbf{P}$, with inter-arrival law $K(\cdot)$, *i.e.*, $\tau_0 = 0$ and $\{\tau_i - \tau_{i-1}\}_{i \in \mathbb{N}}$ is a sequence of IID integer-valued random variables such that

$$\mathbf{P}(\tau_1 = n) =: K(n) \overset{n\to\infty}{\sim} \frac{C_K}{n^{1+\alpha}}, \tag{3.1}$$

with $C_K > 0$ and $\alpha > 0$. We require that $K(\cdot)$ is a probability on $\mathbb{N}$, which amounts to assuming that the renewal process is recurrent. We require also that $K(n) > 0$ for every $n \in \mathbb{N}$, but this is inessential and it is just meant to avoid making a certain number of remarks and small detours in the proofs to take care of this point.

As in Section 2, $\omega := \{\omega_1, \omega_2, \ldots\}$ denotes a sequence of IID standard Gaussian random variables. For a given system size $N \in \mathbb{N}$, coupling parameters $h \in \mathbb{R}$, $\beta \ge 0$ and a given disorder realization ω the partition function of the model is defined by

$$Z_{N,\omega} := \mathbf{E}\left[e^{\sum_{n=1}^N (\beta\omega_n + h - \beta^2/2)\delta_n} \delta_N \right], \tag{3.2}$$

where $\delta_n := \mathbf{1}_{\{n\in\tau\}}$, while the quenched free energy is

$$\textsc{f}(\beta,h) := \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\log Z_{N,\omega}, \tag{3.3}$$

(we use the same notation as for the hierarchical model, since there is no risk of confusion). Like for the hierarchical model, the limit exists and is non-negative [18, Ch. 4], and one defines the critical point $h_c(\beta)$ for a given $\beta \geq 0$ exactly as in (2.6). Again, one notices that the annealed free energy, *i.e.*, the limit of $(1/N)\log \mathbb{E}Z_{N,\omega}$, is nothing but $\textsc{f}(0,h)$, so that the annealed critical point is just $h_c(0)$.

Remark 3.1. With respect to most of the literature, our definition of the model is different (but of course completely equivalent) in that usually the partition function is defined as in (3.2) with $h-\beta^2/2$ replaced simply by h.

The annealed (or pure) model can be exactly solved and in particular it is well known [18, Th. 2.1] that, if $\alpha \neq 1$, there exists a positive constant c_K (which depends on $K(\cdot)$) such that

$$\textsc{f}(0,h) \overset{h\searrow 0}{\sim} c_K h^{\max(1,1/\alpha)}. \tag{3.4}$$

In the case $\alpha = 1$, (3.4) has to be modified in that the right-hand side becomes $\phi(1/h)h$ for some slowly-varying function $\phi(\cdot)$ which vanishes at infinity [18, Th. 2.1]. In particular, note that $h_c(0) = 0$ so that $h_c(\beta) \geq 0$ by Jensen's inequality, exactly like for the hierarchical model.

Remark 3.2. The assumption of recurrence for τ, *i.e.*, $\sum_{n\in\mathbb{N}} K(n) = 1$, is by no means a restriction. In fact, as it has been observed several times in the literature, if $\Sigma_K := \sum_{n\in\mathbb{N}} K(n) < 1$ one can define $\widetilde{K}(n) := K(n)/\Sigma_K$, and of course the renewal τ with law $\widetilde{\mathbf{P}}(\tau_1 = n) = \widetilde{K}(n)$ is recurrent. Then, it is immediate to realize from definition (3.3) that

$$\textsc{f}(\beta,h) = \widetilde{\textsc{f}}(\beta, h+\log\Sigma_K), \tag{3.5}$$

$\widetilde{\textsc{f}}$ being the free energy of the model defined as in (3.2)-(3.3) but with $\mathbf{P}$ replaced by $\widetilde{\mathbf{P}}$. In particular, $h_c^a(\beta) = -\log\Sigma_K$. This observation allows to apply Theorem 3.5 below, for instance, to the case where τ is the set of returns to the origin of a symmetric, finite-variance random walk on $\mathbb{Z}^3$ (pinning of a directed polymer in dimension $(3+1)$): indeed, in this case (3.1) holds with $\alpha = 1/2$. For more details on this issue we refer to [18, Ch. 1].

3.1. **Relevance or irrelevance of disorder.** Like for the hierarchical model, the question whether $h_c(\beta)$ coincides or not with $h_c(0)$ for β small has been recently solved, *except in the marginal case* $\alpha = 1/2$:

Theorem 3.3. *If $0 < \alpha < 1/2$, there exists $\beta_0 > 0$ such that $h_c(\beta) = 0$ for every $0 \leq \beta \leq \beta_0$. If $\alpha = 1/2$, there exists a constant $c_4 > 0$ such that for $\beta \leq 1$*

$$h_c(\beta) \leq \exp(-c_4/\beta^2). \tag{3.6}$$

If $\alpha > 1/2$, $h_c(\beta) > 0$ for every $\beta > 0$ and, if in addition $\alpha \neq 1$, there exists a constant $c_5 > 0$ such that if $\beta \leq 1$

$$c_5\beta^{\max(2\alpha/(2\alpha-1),2)} \leq h_c(\beta) \leq c_5^{-1}\beta^{\max(2\alpha/(2\alpha-1),2)}. \tag{3.7}$$

If $\alpha = 1$ there exist a constant $c_6 > 0$ and a slowly varying function $\psi(\cdot)$ vanishing at infinity such that for $\beta \leq 1$

$$c_6\beta^2\psi(1/\beta) \leq h_c(\beta) \leq c_6^{-1}\beta^2\psi(1/\beta). \tag{3.8}$$

The results for $\alpha \leq 1/2$, together with the critical point upper bounds for $\alpha > 1/2$, have been proven in [2], and then in [31]; the lower bounds on the critical point for $\alpha > 1/2$ have been proven in [10] (the result in [10] is slightly weaker than what we state here and the case $\alpha = 1$ was not treated) and then in [3] (with the full result cited here).

The case $\alpha = 0$ has also been considered, but in that case (3.1) has to be replaced by $K(n) = L(n)/n$, with $L(\cdot)$ a function varying slowly at infinity and such that $\sum_{n\in\mathbb{N}} K(n) = 1$. For instance, this corresponds to the case where τ is the set of returns to the origin of a symmetric random walk on $\mathbb{Z}^2$. In this case, it has been shown in [4] that quenched and annealed critical points coincide for every value of $\beta \geq 0$.

Remark 3.4. Let us recall also that it is proven in [21] that, for every $\alpha > 0$, we have

$$\textsc{f}(\beta, h) \;\leq\; \frac{1+\alpha}{2\beta^2}\,(h - h_c(\beta))^2, \tag{3.9}$$

for all $\beta > 0, h > h_c(\beta)$: this means that when $\alpha > 1/2$ disorder is relevant also in the sense that it changes the free-energy critical exponent (*cf.* (3.4)). The analogous result for the hierarchical model, with $(1+\alpha)$ replaced by some constant $c(B)$ in (3.9), is proven in [28].

In the present work we prove the following:

Theorem 3.5. *Assume that* (3.1) *holds with* $\alpha = 1/2$. *For every* $\beta_0 > 0$ *there exists a constant* $0 < c_7 := c_7(\beta_0) < \infty$ *such that for* $\beta \leq \beta_0$

$$h_c(\beta) \;\geq\; e^{-c_7/\beta^4}. \tag{3.10}$$

4. Marginal relevance of disorder: the hierarchical case

4.1. **Preliminaries: a Galton-Watson representation for X_n.** One can give an expression for X_n which is analogous to that of the partition function (3.2) of the non-hierarchical model, and which is more practical for our purposes. This involves a Galton-Watson tree [26] describing the successive offsprings of one individual. The offspring distribution concentrates on 0 (with probability $(B-1)/B$) and on 2 (with probability $1/B$). So, at a given generation, each individual that is present has either no descendant or two descendants, and this independently of any other individual of the generation. This branching procedure directly maps to a random tree (see Figure 2): the law of such a branching process up to generation n (the first individual is at generation 0) or, analogously, the law of the random tree from the root (level n) up to the leaves (level 0), is denoted by $\mathbf{P}_n$. The individuals that are present at the n^{th} generation are a random subset $\mathcal{R}_n$ of $\{1, \dots, 2^n\}$. We set $\delta_j := \mathbf{1}_{j\in\mathcal{R}_n}$. Note that the mean offspring size is $2/B > 1$, so that the Galton-Watson process is supercritical.

The following procedure on the standard binary graph $\mathcal{T}^{(n)}$ of depth $n+1$ (again, the root is at level n and the leaves, numbered from 1 to 2^n, at level 0) is going to be of help too. Given $\mathcal{I} \subset \{1, \dots, 2^n\}$, let $\mathcal{T}^{(n)}_{\mathcal{I}}$ be the subtree obtained from $\mathcal{T}^{(n)}$ by deleting all edges except those which lead from leaves $j \in \mathcal{I}$ to the root. Note that, with the offspring distribution we consider, in general $\mathcal{T}^{(n)}_{\mathcal{I}}$ is not a realization of the n-generation Galton-Watson tree (some individuals may have just one descendant in $\mathcal{T}^{(n)}_{\mathcal{I}}$, see Figure 2).

Let $v(n, \mathcal{I})$ be the number of nodes in $\mathcal{T}^{(n)}_{\mathcal{I}}$, with the convention that leaves are not counted as nodes, while the root is.

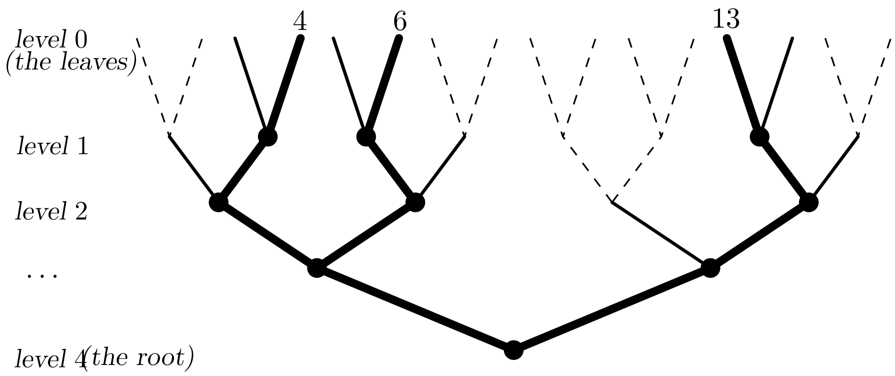

FIGURE 2. The thick solid lines in the figure form the tree $\mathcal{T}^{(4)}_{\{4,6,13\}}$, which is a subtree of the binary tree $\mathcal{T}^{(n)}$ ($n=4$). Note that $\mathcal{T}^{(4)}_{\{4,6,13\}}$ is *not* a possible realization of the Galton-Watson tree, while it becomes so if we complete it by adding the thin solid lines. At level 0 there are the leaves; the nodes of $\mathcal{T}^{(4)}_{\{4,6,13\}}$ are marked by dots. $\mathcal{T}^{(4)}_{\{4,6,13\}}$ contains $v(4,\{4,6,13\})=9$ nodes. In terms of Galton-Watson offsprings, for the (*completed*) trajectory above $\mathcal{R}_4=\{3,4,5,6,13,14\}$. Moreover, computing the average in (4.2) means computing the probability that the realization of the Galton-Watson tree contains $\mathcal{T}^{(n)}_{\mathcal{I}}$ as a subset: but this simply means requiring that the individuals at the nodes of $\mathcal{T}^{(n)}_{\mathcal{I}}$ have two children and the expression (4.2) becomes clear.

Proposition 4.1. *For every $n \geq 0$ we have*

$$X_n \;=\; \mathbf{E}_n\left[e^{\sum_{i=1}^{2^n}(\beta\omega_i+h-\beta^2/2)\delta_i}\right]. \tag{4.1}$$

For every $n\geq 0$ and $\mathcal{I}\subset\{1,\ldots,2^n\}$, one has

$$\mathbf{E}_n\left[\prod_{i\in\mathcal{I}}\delta_i\right] \;=\; B^{-v(n,\mathcal{I})}. \tag{4.2}$$

In particular, $\mathbf{E}_n[\delta_i]=B^{-n}$ for every $i=1,\ldots,2^n$, i.e., the Green function is constant throughout the system.

Proof of Proposition 4.1. The right-hand side in (4.1) for $n=0$ is equal to $\exp(h-\beta^2/2+\beta\omega_1)$. Moreover, at the $(n+1)^{\text{th}}$ generation the branching process either contains only the initial individual (with probability $(B-1)/B$) or the initial individual has two children, which we may look at as initial individuals of two independent Galton-Watson trees containing n new generations. We therefore have that the basic recursion (2.1) is satisfied.

The second fact, (4.2), is a direct consequence of the definitions (see also the caption of Figure 2). □

Remark 4.2. The representation we have introduced in this section shows in particular that $\mathbb{E}X_n$ is just the generating function of $|\mathcal{R}_n|$ and the free energy $\textsc{f}(0,h)$ is therefore a natural quantity for the Galton-Watson process: and in fact $1/\alpha$ (α given in (2.8)) appears in the original works on branching processes by T. E. Harris (of course not to be confused with A. B. Harris, who proposed the disorder relevance criterion on which we are focusing in this work).

4.2. **The proof of Theorem 2.3.** While the discussion of the previous section is valid for every $B \in (1,2)$, now we have to assume $B = B_c = \sqrt{2}$. However some of the steps are still valid in general and we are going to replace B with B_c only when it is really needed. The proof is split into three subsections: the first introduces the fractional moment method and reduces the statement we want to prove, which is a statement on the limit $n \to \infty$ behavior of X_n, to finite-n estimates. The estimates are provided in the second and third subsection.

The fractional moment method. Let $U_n^{(i)}$ denote the quantity $[R_n^{(i)} - (B-1)]_+$ where $[x]_+ = \max(x,0)$. Using the inequality

$$[rs + r + s]_+ \le [r]_+[s]_+ + [r]_+ + [s]_+, \tag{4.3}$$

which holds whenever $r, s \ge -1$, it is easy to check that (2.1) implies

$$U_{n+1}^{(i)} \le \frac{U_n^{(2i-1)}U_n^{(2i)} + (U_n^{(2i-1)} + U_n^{(2i)})(B-1)}{B}. \tag{4.4}$$

Given $0 < \gamma < 1$, we define $A_n := \mathbb{E}([X_n - (B-1)]_+^\gamma)$. From (4.4) above and by using the fractional inequality

$$\left(\sum a_i\right)^\gamma \le \sum a_i^\gamma, \tag{4.5}$$

which holds whenever $a_i \ge 0$, we derive

$$A_{n+1} \le \frac{A_n^2 + 2(B-1)^\gamma A_n}{B^\gamma}. \tag{4.6}$$

One readily sees now that, if there exists some integer k such that

$$A_k \,<\, B^\gamma - 2(B-1)^\gamma, \tag{4.7}$$

then A_n tends to zero as n tends to infinity (this statement is easily obtain by studying the fixed points of the function $x \mapsto (x^2 + 2(B-1)^\gamma x)/B^\gamma$). On the other hand,

$$\mathbb{E}\left[X_n^\gamma\right] \le \mathbb{E}\left([X_n - (B-1)]_+ + (B-1)\right)^\gamma \le (B-1)^\gamma + A_n, \tag{4.8}$$

and therefore (4.7) implies that $\textsc{f}(\beta,h) = 0$ since, by Jensen inequality, we have

$$\frac{1}{2^n}\mathbb{E}\log X_n \,\le\, \frac{1}{2^n\gamma}\log \mathbb{E}\left[X_n^\gamma\right]. \tag{4.9}$$

Note that, to establish $\textsc{f}(\beta,h) = 0$, it suffices to prove that $\limsup_n 2^{-n}\log A_n \le 0$, hence our approach yields a substantially stronger piece of information, *i.e.* that the fractional moment A_n does go to zero.

In order to find a k such that (4.7) holds we introduce a new probability measure $\widetilde{\mathbb{P}}$ (which is going to depend on k) such that $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, that is mutually absolutely continuous. By Hölder's inequality applied for $p = 1/\gamma$ and $q = 1/(1-\gamma)$ we have

$$A_k \,=\, \widetilde{\mathbb{E}}\left[\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\left[X_k - (B-1)\right]_+^\gamma\right] \le \left(\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\right)^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\gamma}\left(\widetilde{\mathbb{E}}\left[[X_k - (B-1)]_+\right]\right)^\gamma, \tag{4.10}$$

and a sufficient condition for (4.7) is therefore that

$$\widetilde{\mathbb{E}}\left[[X_k - (B-1)]_+\right] \,\le\, \left(\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\right)^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\frac{1}{\gamma}}\left(B^\gamma - 2(B-1)^\gamma\right)^{\frac{1}{\gamma}}. \tag{4.11}$$

Let $x_n^{(0)}$ be obtained applying n times the annealed iteration $x \mapsto (x^2+(B-1))/B$ to the initial condition $x_0^{(0)}=0$. One has that $x_n^{(0)}$ approaches monotonically the stable fixed point $(B-1)$. Since the coefficients in the iteration (2.1) are positive, one has for every h, β, ω that $X_n \ge x_n^{(0)} \overset{n\to\infty}{\nearrow} B-1$ (this is a deterministic bound) and therefore, for any given $\zeta>0$, one can find an integer n_ζ such that if $n \ge n_\zeta$ we have

$$\widetilde{\mathbb{E}}\left[[X_n-(B-1)]_+\right] \le \widetilde{\mathbb{E}}\left[X_n-(B-1)\right]+\frac{\zeta}{4}. \tag{4.12}$$

Moreover, since $(B^\gamma-2(B-1)^\gamma)^{\frac{1}{\gamma}}-(2-B) \overset{\gamma\nearrow 1}{\sim} -c_B(1-\gamma)$ for some $c_B>0$, one can find $\gamma=\gamma_\zeta$ such that $(B^\gamma-2(B-1)^\gamma)^{\frac{1}{\gamma}} \ge 2-B-\zeta/4$. At this point, if $\gamma=\gamma_\zeta$, $k \ge n_\zeta$ and if $\widetilde{\mathbb{P}}$ is such that

$$\left(\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\right)^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\frac{1}{\gamma}} \ge 1-\frac{\zeta}{4}. \tag{4.13}$$

(recall that $\widetilde{\mathbb{P}}$ depends on k) and $\widetilde{\mathbb{E}}[X_k] \le 1-\zeta$ then (4.11) is satisfied and $\textsc{f}(\beta,h)=0$.

We sum up what we have obtained:

Lemma 4.3. *Let $\zeta>0$ and choose $\gamma(=\gamma_\zeta)$ and n_ζ as above. If there exists $k \ge n_\zeta$ and a probability measure $\widetilde{\mathbb{P}}$ (such that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent probabilities) such that*

$$\left(\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\right)^{\frac{\gamma}{1-\gamma}}\right]\right)^{1-\frac{1}{\gamma}} \ge 1-\frac{\zeta}{4}, \tag{4.14}$$

and

$$\widetilde{\mathbb{E}}[X_k] \le 1-\zeta, \tag{4.15}$$

then the free energy is equal to zero.

The change of measure. In order to use wisely the result of the previous section, we have to find a measure $\widetilde{\mathbb{E}} := \widetilde{\mathbb{E}}_n$ on the environment which is, in a sense, close to $\mathbb{E}$ (*cf.* (4.14)), and that lowers significantly the expectation of X_n. In [19] we introduced the idea of changing the mean of the ω-variables, while keeping their IID character. This strategy was enough to prove disorder relevance for $B<B_c$, but it is not effective in the marginal case $B=B_c$ we are considering here. Here, instead, we choose to introduce *weak, long range* negative correlations between the different ω_i without changing the laws of the 1-dimensional marginals. As it will be clear, the covariance structure we choose reflects the hierarchical structure of the model we are considering.

In the sequel we take $h \ge h_c(0)=0$.

We define $\widetilde{\mathbb{P}}_n$ by stipulating that the variables $\omega_i, i>2^n$, are still IID standard Gaussian independent of $\omega_1,\ldots,\omega_{2^n}$, while $\omega_1,\ldots,\omega_{2^n}$ are Gaussian, centered, and with covariance matrix

$$C := I-\varepsilon V, \tag{4.16}$$

where I is the $2^n \times 2^n$ identity matrix, $\varepsilon>0$ and V is a symmetric $2^n \times 2^n$ matrix with zero diagonal terms and with positive off-diagonal terms (ε and V will be specified in a moment).

The choice $V_{ii}=0$ implies of course $\mathrm{Trace}(V)=0$, and we are also going to impose that the Hilbert-Schmidt norm of V verifies $\|V\|^2 := \sum_{i,j} V_{i,j}^2 = \mathrm{Trace}(V^2)=1$. This in

particular implies that C is positive definite (so that $\widetilde{\mathbb{P}}_n$ exists!) as soon as $\varepsilon < 1$: this is because $\|V\|$, being a matrix norm, dominates the spectral radius of V.

Now, still without choosing V explicitly, we compute a lower bound for the left–hand side of (4.14). The mutual density of $\widetilde{\mathbb{P}}_n$ and $\mathbb{P}$ is

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_n}{\mathrm{d}\mathbb{P}}(\omega) \,=\, \frac{e^{-1/2((C^{-1}-I)\omega,\omega)}}{\sqrt{\det C}}, \tag{4.17}$$

with the notation $(Av, v) := \sum_{1\le i,j\le 2^n} A_{ij}v_iv_j$, and therefore a straightforward Gaussian computation gives

$$\left(\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}_n}\right)^{\gamma/(1-\gamma)}\right]\right)^{1-1/\gamma} \,=\, \frac{(\det[I-(\varepsilon/(1-\gamma))V])^{(1-\gamma)/(2\gamma)}}{(\det C)^{1/(2\gamma)}}. \tag{4.18}$$

If we want to prove a lower bound of the type (4.14), a necessary condition is of course that the numerator in (4.18) is positive: this is ensured by requiring $\varepsilon < 1-\gamma$. For the next computation we are going to require also that $\varepsilon/(1-\gamma) \le 1/2$: we are going in fact to use that $\log(1+x) \ge x - x^2$ if $x \ge -1/2$, and $\mathrm{Trace}(V) = 0$ to obtain that

$$\begin{aligned}\det\left[I-(\varepsilon/(1-\gamma))V\right] \,&=\, \exp\left(\mathrm{Trace}(\log(I-(\varepsilon/(1-\gamma))V))\right)\\ &\ge\, \exp\left(-\frac{\varepsilon^2}{(1-\gamma)^2}\|V\|^2\right),\end{aligned} \tag{4.19}$$

while $\log(1+x) \le x$ and the traceless character of V directly imply $\det C \le 1$ so that finally

$$\left(\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}_n}\right)^{\gamma/(1-\gamma)}\right]\right)^{1-1/\gamma} \,\ge\, \exp\left(-\frac{\varepsilon^2}{2\gamma(1-\gamma)}\right). \tag{4.20}$$

Next, we estimate the expected value of X_n under the modified measure: from (4.1) we see that

$$\begin{aligned}\widetilde{\mathbb{E}}_n X_n \,&=\, \mathbf{E}_n\left[e^{(h-(\beta^2/2))\sum_{i=1}^{2^n}\delta_i}\,\widetilde{\mathbb{E}}_n e^{\sum_{i=1}^{2^n}\beta\omega_i\delta_i}\right]\\ &=\, \mathbf{E}_n\left[e^{-\varepsilon(\beta^2/2)(V\delta,\delta)+\sum_{i=1}^{2^n}h\delta_i}\right] \,\le\, e^{2^n h}\,\mathbf{E}_n\left[e^{-\varepsilon(\beta^2/2)(V\delta,\delta)}\right].\end{aligned} \tag{4.21}$$

Finally we choose V. From (4.21), it is not hard to guess that the most convenient choice, subject to the constraint $\|V\|^2 = 1$, is

$$V_{ij} \,=\, \mathbf{E}_n[\delta_i\delta_j]\Big/\sqrt{\sum_{1\le i\ne j\le 2^n}(\mathbf{E}_n[\delta_i\delta_j])^2}, \tag{4.22}$$

for $i \ne j$, while we recall that $V_{ii} = 0$. The normalization in (4.22) can be computed with the help of Proposition 4.1:

$$\sum_{1\le i\ne j\le 2^n}(\mathbf{E}_n[\delta_i\delta_j])^2 = 2^n\sum_{1<j\le 2^n}(\mathbf{E}_n[\delta_1\delta_j])^2 = 2^n\sum_{1\le a\le n}\frac{2^{a-1}}{B_c^{2(n+(a-1))}} = n. \tag{4.23}$$

In the second equality, we used the fact that there are 2^{a-1} values of $1 < j \le 2^n$ such that the two branches of the tree $\mathcal{T}^{(n)}_{\{1,j\}}$ join at level a (*cf.* the notations of Section 4.1), and such tree contains $n+a-1$ nodes.

As a side remark, note that if $B_c < B < 2$ (irrelevant disorder regime) the left-hand side of (4.23) instead goes to zero with n, while for $1 < B < B_c$ (relevant disorder regime) it diverges exponentially with n.

So, in the end, our choice for V is:

$$V_{ij} = \begin{cases} \mathbf{E}_n[\delta_i \delta_j]/\sqrt{n} & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j. \end{cases} \tag{4.24}$$

Checking the conditions of Lemma 4.3. To conclude the proof of Theorem 2.3 we have to show that if $\beta \le \beta_0$ and $h \le \exp(-c_3/\beta^4)$ (and provided that $c_3 = c_3(\beta_0)$ is chosen large enough) the conditions of Lemma 4.3 are satisfied. The main point is therefore to estimate the expectation of X_n under $\widetilde{\mathbb{P}}_n$.

Recalling that (*cf.* (4.21))

$$\widetilde{\mathbb{E}}_n X_n \le \mathbf{E}_n \left[e^{-(\beta^2/2)\varepsilon \sum_{1\le i\neq j\le 2^n} \delta_i \delta_j \frac{\mathbf{E}_n[\delta_i\delta_j]}{\sqrt{n}}} \right] e^{2^n h}, \tag{4.25}$$

we define

$$Y_n := \sum_{1\le i\neq j\le 2^n} \delta_i \delta_j \frac{\mathbf{E}_n[\delta_i \delta_j]}{n}. \tag{4.26}$$

Thanks to (4.23), we know that $\mathbf{E}_n(Y_n) = 1$, so that the Paley-Zygmund inequality gives

$$\mathbf{P}_n\left(Y_n \ge 1/2\right) \;=\; \mathbf{P}_n\left(Y_n \ge (1/2)\mathbf{E}_n(Y_n)\right) \;\ge\; \frac{(\mathbf{E}_n(Y_n))^2}{4\,\mathbf{E}_n(Y_n^2)} = \frac{1}{4\,\mathbf{E}_n(Y_n^2)}. \tag{4.27}$$

We need therefore the following estimate, which will be proved at the end of the section:

Lemma 4.4. *We have:*

$$(1 \le)\ \mathcal{K} := \sup_n \mathbf{E}_n[Y_n^2] < \infty. \tag{4.28}$$

Together with (4.27) this implies

$$\mathbf{P}_n[Y_n \ge 1/2] \;\ge\; \frac{1}{4\mathcal{K}}, \tag{4.29}$$

so that, for all $n \ge 0$,

$$\begin{aligned} \mathbf{E}_n\left[e^{-(\beta^2/2)\varepsilon \sum_{1\le i\neq j\le 2^n} \delta_i\delta_j \frac{\mathbf{E}_n[\delta_i\delta_j]}{\sqrt{n}}} \right] \;&=\; \mathbf{E}_n\left[e^{-\frac{\sqrt{n}\beta^2\varepsilon}{2} Y_n} \right] \\ &\le\; 1 - \frac{1}{4\mathcal{K}}\left(1 - 4\mathcal{K}\exp\left(-\frac{\sqrt{n}\beta^2\varepsilon}{4}\right)\right). \end{aligned} \tag{4.30}$$

We fix $\zeta := 1/(40\mathcal{K})$ and we choose $\gamma = \gamma_\zeta$ (*cf.* Lemma 4.3) and ε in (4.16) small enough so that (*cf.* (4.20))

$$\left[\mathbb{E}\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}_n} \right)^{\gamma/(1-\gamma)} \right]^{1-1/\gamma} \;\ge\; \exp\left(-\frac{\varepsilon^2}{2\gamma(1-\gamma)} \right) \ge 1 - \frac{\zeta}{4}. \tag{4.31}$$

Then one can check with the help of (4.30) that for $n \ge 50\mathcal{K}/(\beta^4\varepsilon^2)$,

$$\mathbf{E}_n\left[e^{-(\beta^2/2)\varepsilon \sum_{1\le i\neq j\le 2^n} \delta_i\delta_j \frac{\mathbf{E}_n[\delta_i\delta_j]}{\sqrt{n}}} \right] \;\le\; 1 - 3\zeta. \tag{4.32}$$

We choose $n = n_\beta$ in $\left[\frac{50\mathcal{K}}{\beta^4\varepsilon^2}, \frac{50\mathcal{K}}{\beta^4\varepsilon^2}+1\right)$ and $h = \zeta 2^{-n}$. If ε has been chosen small enough above (how small, depending only on β_0), this guarantees that $n \geq n_\zeta$, where n_ζ was defined just before Lemma 4.3. Injecting (4.32) in (4.25) finally gives

$$\widetilde{\mathbb{E}}[X_n] \leq (1-3\zeta)e^{\zeta} \leq 1-\zeta. \tag{4.33}$$

The two conditions of Lemma 4.3 are therefore verified, which ensures that the free energy is zero for this value of h. In conclusion, for every $\beta \leq \beta_0$ we have proven that

$$h_c(\beta) \;\geq\; \zeta\, 2^{-n_\beta} \;\geq\; \frac{1}{80\mathcal{K}} \exp\left(-\frac{50\mathcal{K}\log 2}{\beta^4\varepsilon^2}\right), \tag{4.34}$$

for some $\varepsilon = \varepsilon(\beta_0)$ sufficiently small but independent of β. □

Proof of Lemma 4.4. We have

$$\mathbf{E}_n(Y_n^2) = \frac{1}{n^2} \sum_{1\leq i\neq j\leq 2^n} \sum_{1\leq k\neq l\leq 2^n} \mathbf{E}_n[\delta_i\delta_j]\mathbf{E}_n[\delta_k\delta_l]\mathbf{E}_n[\delta_i\delta_j\delta_k\delta_l]. \tag{4.35}$$

We will consider only the contribution coming from the terms such that $i \neq k, l$ and $j \neq k, l$. The remaining terms can be treated similarly and their global contribution is easily seen to be exponentially small in n. (For instance, when $i = k$ and $j = l$ one gets

$$\frac{1}{n^2} \sum_{1\leq i\neq j\leq 2^n} \mathbf{E}_n[\delta_i\delta_j]^3 \;\leq\; \frac{1}{n}\mathbf{E}_n(Y_n) \max_{1\leq i<j\leq 2^n} \mathbf{E}_n[\delta_i\delta_j], \tag{4.36}$$

which is exponentially small in n, in view of Theorem 4.1.)

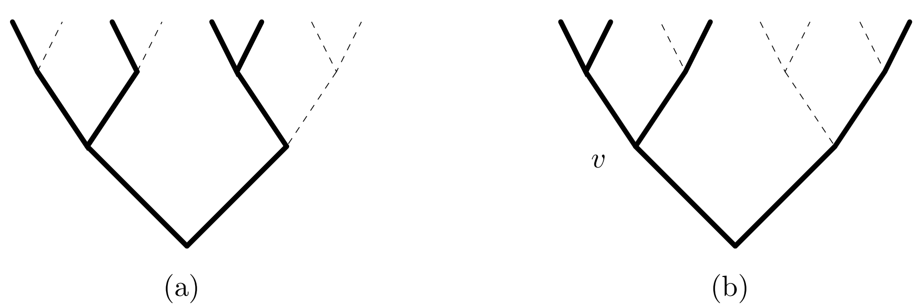

FIGURE 3. The two different possible topologies of the tree $\mathcal{T}^{(n)}_{\{i,j,k,l\}}$. Case (b) is understood to include also the trees where the branch which does not bifurcate is the one on the left, or where the sub-branch which bifurcates is the right descendent of the node v. We consider only trees where the four leaves are distinct, since the remaining ones give a contribution to $\mathbf{E}_n(Y_n^2)$ which vanishes for $n \to \infty$.

From now on, therefore, we assume that i, j, k, l are all distinct. Two cases can occur:

(1) the tree $\mathcal{T}^{(n)}_{\{i,j,k,l\}}$ (it is better to view it here has the backbone tree, not as the Galton-Watson tree, see Figure 2) has two branches, which themselves bifurcate into two sub-branches, *cf.* Fig. 3(a) for an example. We call c the level at which the first bifurcation occurs ($c = n$ in the example of Fig. 3(a)), and a, b the levels at which the two branches bifurcate. One has clearly $1 \leq a < c \leq n$ and $1 \leq b < c \leq n$. All trees of this form can be obtained as follows: first choose a leaf f_1, between 1 and 2^n. Then choose f_2 among the 2^{a-1} possible ones which

join with f_1 at level a, f_3 among the 2^{c-1} which join with f_1 at level c and finally f_4 among the 2^{b-1} which join with f_3 at level b. Clearly we are over-counting the trees (note for example that already in the choice of f_1 and f_2 we are over-counting by a factor 2), but we are only after an *upper bound* for $\mathbf{E}_n(Y_n^2)$ (the same remark applies to case (2) below). We still have to specify how to identify (f_1, f_2, f_3, f_4) with a permutation of (i, j, k, l). When $(f_1, f_2, f_3, f_4) = (i, j, k, l)$ we get the following contribution to (4.35):

$$\frac{1}{n^2} \sum_{1 \le a < c \le n} \sum_{1 \le b < c \le n} \frac{2^{n+a+b+c-3}}{B_c^{n+a+b+c-3} B_c^{n+a-1} B_c^{n+b-1}}, \tag{4.37}$$

where we used Theorem 4.1 to write, *e.g.*, $\mathbf{E}_n[\delta_i \delta_j] = B_c^{-n-a+1}$. Since $B_c = \sqrt{2}$ we can rewrite (4.37) as

$$\frac{1}{\sqrt{2} n^2} \sum_{1 < c \le n} (c-1)^2 2^{-(n-c)/2}, \tag{4.38}$$

which is clearly bounded as n grows.

If instead $(f_1, f_2, f_3, f_4) = (i, k, j, l)$ or $(f_1, f_2, f_3, f_4) = (i, k, l, j)$, one gets

$$\frac{1}{n^2} \sum_{1 \le a < c \le n} \sum_{1 \le b < c \le n} \frac{2^{n+a+b+c-3}}{B_c^{n+a+b+c-3} B_c^{n+c-1} B_c^{n+c-1}}, \tag{4.39}$$

which is easily seen to be $O(1/n^2)$.

All the other permutations of (i, j, k, l) give a contribution which equals, by symmetry, one of the three we just considered.

(2) the tree $\mathcal{T}^{(n)}_{\{i,j,k,l\}}$ has two branches: one of them does not bifurcate, the other one bifurcates into two sub-branches, one of which bifurcates into two sub-sub-branches, *cf.* Figure 3(b). Let a_1, a_2, a_3 be the levels where the three bifurcations occur, ordered so that $1 \le a_1 < a_2 < a_3 \le n$. This time, we choose f_1 between 1 and 2^n and then, for $i = 1, 2, 3$, f_{i+1} among the $2^{a_i - 1}$ leaves which join with f_1 at level a_i. If $(f_1, f_2, f_3, f_4) = (i, j, k, l)$ one has in this case

$$\frac{1}{n^2} \sum_{1 \le a_1 < a_2 < a_3 \le n} \frac{2^{n+a_1+a_2+a_3-3}}{B_c^{n+a_1+a_2+a_3-3} B_c^{n+a_1-1} B_c^{n+a_3-1}} = \frac{1}{\sqrt{2} n^2} \sum_{1 \le a_1 < a_2 < a_3 \le n} 2^{-(n-a_2)/2}, \tag{4.40}$$

which is $O(1/n)$. Finally, when (f_1, f_2, f_3, f_4) is equal to (i, k, j, l) or to (i, k, l, j) one gets

$$\frac{1}{n^2} \sum_{1 \le a_1 < a_2 < a_3 \le n} \frac{2^{n+a_1+a_2+a_3-3}}{B_c^{n+a_1+a_2+a_3-3} B_c^{n+a_2-1} B_c^{n+a_3-1}} = \frac{1}{\sqrt{2} n^2} \sum_{1 \le a_1 < a_2 < a_3 \le n} 2^{-(n-a_1)/2}, \tag{4.41}$$

which is $O(1/n^2)$.

$\square$

5. Marginal relevance of disorder: the non-hierarchical case

Here we prove Theorem 3.5 and therefore we assume that (3.1) holds with $\alpha = 1/2$. We choose and fix once and for all a $\gamma \in (2/3, 1)$ and set for $h > 0$

$$k := k(h) := \left\lfloor \frac{1}{h} \right\rfloor . \tag{5.1}$$

Remark 5.1. In [10] the choice $k(h) = \lfloor 1/\mathrm{F}(0,h) \rfloor$ was made and it corresponds to choosing $k(h)$ equal to the correlation length of the annealed system. In our case $1/\mathrm{F}(0,h) \overset{h \searrow 0}{\sim} 1/(c_K h^2)$ (*cf.* (3.4)) and therefore (5.1) may look surprising. However, there is nothing particularly deep behind: for $\alpha = 1/2$, due to the fact that we have to prove delocalization for $h \le \exp(-c_7/\beta^4)$, choosing $k(h)$ that diverges for small h like $1/h$ instead of $1/h^2$ just leads to choosing c_7 different by a factor 2 (and we do not track the precise value of constants). We take this occasion to stress that it is practical to work always with sufficiently large values of $k(h)$, and this can be achieved by choosing c_7 sufficiently large.

We divide $\mathbb{N}$ into blocks

$$B_i := \{(i-1)k + 1, (i-1)k + 2, \ldots, ik\} \text{ with } i = 1, 2, \ldots . \tag{5.2}$$

From now on we assume that (N/k) is integer, and of course it is also the number of blocks contained in the interval $\{1, \ldots, N\}$.

We define, in analogy with the hierarchical case,

$$A_N := \mathbb{E}\left(Z_{N,\omega}^{\gamma} \right), \tag{5.3}$$

and we note that, as in (4.9), Jensen's inequality implies that a sufficient condition for $\mathrm{F}(\beta, h) = 0$ is that A_N does not diverge when $N \to \infty$. Therefore, our task is to show that for every $\beta_0 > 0$ we can find $c_7 > 0$ such that for every $\beta \le \beta_0$ and h such that

$$0 < h \le \exp(-c_7/\beta^4), \tag{5.4}$$

one has that $\sup_N A_N < \infty$.

5.1. **Decomposition of $Z_{N,\omega}$ and change of measure.** The first step is a decomposition of the partition function similar to that used in [33], which is a refinement of the strategy employed in [10]. For $0 < i \le j$ we let $Z_{i,j} := Z_{(j-i),\theta^i\omega}$, with $(\theta^i \omega)_a := \omega_{i+a}, a \in \mathbb{N}$, *i.e.*, $\theta^i\omega$ is the result of the application to ω of a shift by i units to the left. We decompose $Z_{N,\omega}$ according to the value of the first point (n_1) of τ after 0, the last point (j_1) of τ not exceeding $n_1 + k - 1$, then the first point (n_2) of τ after j_1, and so on. We call i_r the index of the block in which n_r falls, and $\ell := \max\{r : n_r \le N\}$, see Figure 4. Due to the constraint $N \in \tau$, one has always $i_\ell = (N/k)$.

In formulas:

$$Z_{N,\omega} = \sum_{\ell=1}^{N/k} \sum_{i_0 := 0 < i_1 < \ldots < i_\ell = N/k} \widehat{Z}_\omega^{(i_1,\ldots,i_\ell)}, \tag{5.5}$$

where

$$\widehat{Z}_\omega^{(i_1,\ldots,i_\ell)} := \sum_{n_1 \in B_{i_1}} \sum_{j_1 = n_1}^{n_1+k-1} \sum_{\substack{n_2 \in B_{i_2}: \\ n_2 \ge n_1 + k}} \sum_{j_2 = n_2}^{n_2+k-1} \cdots \sum_{\substack{n_{\ell-1} \in B_{i_{\ell-1}}: \\ n_{\ell-1} \ge n_{\ell-2}+k}} \sum_{j_{\ell-1}=n_{\ell-1}}^{n_{\ell-1}+k-1} \sum_{\substack{n_\ell \in B_{N/k}: \\ n_\ell \ge n_{\ell-1}+k}}$$
$$z_{n_1} K(n_1) Z_{n_1,j_1} z_{n_2} K(n_2 - j_1) Z_{n_2,j_2} \ldots z_{n_\ell} K(n_\ell - j_{\ell-1}) Z_{n_\ell,N}, \tag{5.6}$$

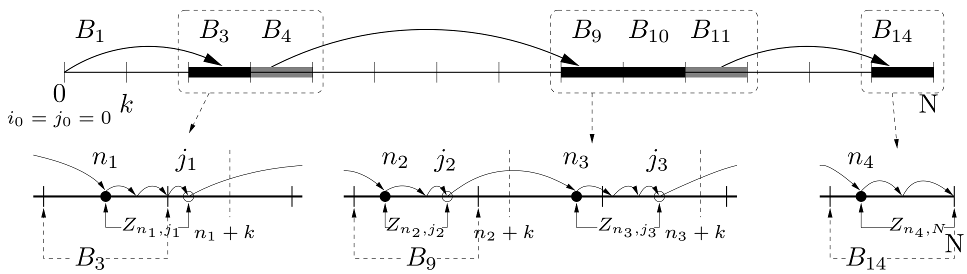

Figure 4. A typical configuration which contributes to $\widehat{Z}_\omega^{(i_1,\dots,i_\ell)}$. In this example we have $N/k = 14$, $\ell = 4$, $i_1 = 3$, $i_2 = 9$, $i_3 = 10$ and $i_4 = N/k = 14$ (by definition $i_\ell = N/k$, *cf.* (5.5)). Contact points are only in black and grey blocks: the blocks B_{i_j}, $j = 1, \dots, \ell$ are black and they contain one (and only one) point n_i. To the right of a black block there is either another black block or a grey block (except for the last black block, B_{i_ℓ}, that contains the end-point N of the system). The bottom part of the figure zooms on black and grey blocks. We see that to the right of n_i (big black dots) there are renewal points before $n_i + k$; for $i < \ell$, j_i is the rightmost one and it is marked by a big empty dot (even if it is not the case in the figure, it may happen that there is none: in that case $j_i = n_i$). Therefore, between empty dots and black dots there is no contact point (the origin should be considered an empty dot too). Note that j_i can be in B_i, as it is the case for j_2, or in B_{i+1}, as it is the case for j_1 and j_3. Going back to the figure on top, we observe that the set M of (5.9) is $\{3, 4, 9, 10, 11, 14\}$, that is the collection of black and grey blocks. We point out that it may happen that a grey block contains no point, but it is convenient for us to treat grey blocks as if they always contained contact points. It is only to the charges ω in black and grey blocks that we apply the change-of-measure argument that is crucial for our proof.

and $z_n := e^{\beta\omega_n + h - \beta^2/2}$.

Then, from inequality (4.5), we have

$$A_N \le \sum_{\ell=1}^{N/k} \sum_{i_0:=0<i_1<\dots<i_\ell=N/k} \mathbb{E}\left[(\widehat{Z}_\omega^{(i_1,\dots,i_\ell)})^\gamma\right], \tag{5.7}$$

and, as in (4.10), we apply Hölder's inequality to get

$$\mathbb{E}\left[\left(\widehat{Z}_\omega^{(i_1,\dots,i_\ell)}\right)^\gamma\right] = \widetilde{\mathbb{E}}\left[\left(\widehat{Z}_\omega^{(i_1,\dots,i_\ell)}\right)^\gamma \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}(\omega)\right] \le \left(\widetilde{\mathbb{E}}\widehat{Z}_\omega^{(i_1,\dots,i_\ell)}\right)^\gamma \left(\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\right)^{\gamma/(1-\gamma)}\right]\right)^{1-\gamma}. \tag{5.8}$$

The new law $\widetilde{\mathbb{P}} := \widetilde{\mathbb{P}}^{(i_1,\dots,i_\ell)}$ will be taken to depend on the set $(i_1, \dots, i_\ell)$. In order to define it, let first of all

$$M := M(i_1, \dots, i_\ell) := \{i_1, i_2, \dots, i_\ell\} \cup \{i_1 + 1, i_2 + 1, \dots, i_{\ell-1} + 1\}. \tag{5.9}$$

Then, we say that under $\widetilde{\mathbb{P}}$ the random vector ω is Gaussian, centered and with covariance matrix

$$\widetilde{\mathbb{E}}(\omega_i\omega_j) = \mathbf{1}_{i=j} - \mathcal{C}_{ij} := \begin{cases} \mathbf{1}_{i=j} - H_{ij} & \text{if there exists } u \in M \text{ such that } i, j \in B_u, \\ \mathbf{1}_{i=j} & \text{otherwise,} \end{cases} \tag{5.10}$$

and

$$H_{ij} := \begin{cases} (1-\gamma)/\sqrt{9\,k(\log k)\ |i-j|} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases} \tag{5.11}$$

Note that all the $\mathcal{C}_{ij}$'s are non-negative. It is immediate to check that the $k \times k$ symmetric matrix $\widehat{H} := \{H_{ij}\}_{i,j=1}^{k}$ satisfies

$$\|\widehat{H}\| := \sqrt{\sum_{i,j=1}^{k} H_{ij}^2} \leq \frac{1-\gamma}{2}, \tag{5.12}$$

for k sufficiently large. In words: ω_n's in different blocks are independent; in blocks B_u with $u \notin M$ they are just IID standard Gaussian random variables, while if $u \in M$ then the random vector $\{\omega_n\}_{n \in B_u}$ has covariance matrix $I - \widehat{H}$, where I is the $k \times k$ identity matrix. Note that, since $\|\widehat{H}\|$ dominates the spectral radius of $\widehat{H}$, (5.12) guarantees that $I - \widehat{H}$ is positive definite (and also that $I - (1-\gamma)^{-1}\widehat{H}$ is positive definite, that will be needed just below).

The last factor in the right-hand side of (5.8) is easily obtained recalling (4.18) and independence of the ω_n's in different blocks, and one gets

$$\left(\widetilde{\mathbb{E}}\left[\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\right)^{\gamma/(1-\gamma)}\right]\right)^{1-\gamma} = \left(\frac{\det(I-\widehat{H})}{(\det(I-1/(1-\gamma)\widehat{H}))^{1-\gamma}}\right)^{|M|/2}. \tag{5.13}$$

Since $\widehat{H}$ has trace zero and its (Hilbert-Schmidt) norm satisfies (5.12), one can apply $\det(I-\widehat{H}) \leq \exp(-\mathrm{Trace}(\widehat{H})) = 1$ and (4.19) (with V replaced by $\widehat{H}$ and ε by 1) to get that the right-hand side of (5.13) is bounded above by $\exp(|M|/2)$, which in turn is bounded by $\exp(\ell)$. Together with (5.8) and (5.7), we conclude that

$$A_N \leq \sum_{\ell=1}^{N/k} \sum_{i_0:=0<i_1<\ldots<i_\ell=N/k} e^{\ell}\left[\widetilde{\mathbb{E}}\widehat{Z}_{\omega}^{(i_1,\ldots,i_\ell)}\right]^{\gamma}. \tag{5.14}$$

5.2. **Reduction to a non-disordered model.** We wish to bound the right-hand side of (5.14) with the partition function of a non-disordered pinning model in the delocalized phase, which goes to zero for large N. We start by claiming that

$$\widetilde{\mathbb{E}}\widehat{Z}_{\omega}^{(i_1,\ldots,i_\ell)} \leq \sum_{n_1 \in B_{i_1}} \cdots \sum_{\substack{n_\ell \in B_{N/k}: \\ n_\ell \geq n_{\ell-1}+k}} K(n_1)K(n_2-j_1)\ldots K(n_\ell - j_{\ell-1}) \\ \times U(j_1-n_1)U(j_2-n_2)\ldots U(N-n_\ell), \tag{5.15}$$

where

$$U(n) = c_8 \mathbf{P}(n \in \tau)\mathbf{E}\left[e^{-\beta^2 \sum_{1 \leq i < j \leq n/2} H_{ij}\delta_i\delta_j}\right], \tag{5.16}$$

and c_8 is a positive constant depending only on $K(\cdot)$. This is proven in Appendix A.2. We are also going to make use of:

Lemma 5.2. *There exists $C_2 = C_2(K(\cdot)) < \infty$ such that if, for some $\eta > 0$,*

$$\sum_{j=0}^{k-1} U(j) \leq \eta\sqrt{k} \tag{5.17}$$

and

$$\sum_{j=0}^{k-1}\sum_{n\geq k} U(j)K(n-j)\leq \eta, \tag{5.18}$$

then there exists $C_1 = C_1(\eta, k, K(\cdot))$ *such that the right-hand side of* (5.15) *is bounded above by*

$$C_1\eta^\ell C_2^\ell \prod_{r=1}^{\ell}\frac{1}{(i_r-i_{r-1})^{3/2}}. \tag{5.19}$$

It is important to note that C_2 does not depend on η.

Lemma 5.2 is a small variation on [33, Lemma 3.1], but, both because the model we are considering is somewhat different and for sake of completeness, we give the details of the proof in Appendix A.2.

Now assume that conditions (5.17)-(5.18) are verified for some η. Collecting (5.14), (5.15) and Lemma 5.2, we have then

$$A_N \leq C_1^\gamma \sum_{\ell=1}^{N/k}\sum_{i_0:=0<i_1<\ldots<i_\ell=N/k} (\eta^\gamma\, C_2^\gamma e)^\ell \prod_{r=1}^{\ell}\frac{1}{(i_r-i_{r-1})^{(3/2)\gamma}}. \tag{5.20}$$

In the right-hand side we recognize, apart from the irrelevant multiplicative constant C_1^γ, the partition function of a non-random ($\beta = 0$) pinning model with N replaced by N/k, $K(\cdot)$ replaced by

$$\widehat{K}(n) \;=\; \frac{1}{n^{(3/2)\gamma}}\frac{1}{\sum_{i\geq 1} i^{-(3/2)\gamma}}, \tag{5.21}$$

and h replaced by

$$\widehat{h} := \log\left(\eta^\gamma\, C_2^\gamma e \sum_{n\in\mathbb{N}}\frac{1}{n^{(3/2)\gamma}}\right). \tag{5.22}$$

Note that $\widehat{K}(\cdot)$ is normalized to be a probability measure on $\mathbb{N}$, which is possible since (by assumption) $\gamma > 2/3$, and that it has a power-law tail with exponent $(3/2)\gamma > 1$. Thanks to Lemma A.1 below, one has that the right-hand side of (5.20) tends to zero for $N\to\infty$ whenever

$$\widehat{h} < 0. \tag{5.23}$$

Therefore, if η is so small that (5.23) holds, we can conclude that A_N tends to zero for $N\to\infty$ and therefore $\textsc{f}(\beta,h) = 0$.

The proof of Theorem 3.5 is therefore concluded once we prove

Proposition 5.3. *Fix* $\eta > 0$ *such that* (5.23) *holds. For every* $\beta_0 > 0$ *there exists* $0 < c_7 < \infty$ *such that if* $\beta \leq \beta_0$ *and* $0 < h \leq \exp(-c_7/\beta^4)$*, conditions* (5.17)-(5.18) *are verified.*

Proof of Proposition 5.3. We need to show that the two hypotheses of Lemma 5.2 hold and for this we are going to use the following result:

Lemma 5.4. *Under the law* $\mathbf{P}$*, the random variable*

$$W_L := (\sqrt{L}\log L)^{-1} \sum_{1\le i<j\le L} \delta_i\delta_j/\sqrt{j-i}, \tag{5.24}$$

converges in distribution, as L *tends to* ∞*, to* $c|Z|$ *(*$Z \sim N(0,1)$ *and* c *a positive constant).*

This lemma, the proof of which may be found just below (together with the explicit value of c), directly implies that, if we set $S(a,L) := \mathbf{E}\left[\exp(-aW_L)\right]$, we have $\lim_{a\to\infty}\lim_{L\to\infty} S(a,L) = 0$ and, by the monotonicity of $S(\cdot,L)$, we get

$$\lim_{a,L\to\infty} S(a,L) = 0. \tag{5.25}$$

Let us verify (5.17). Note first of all that (*cf.* (5.16) and (5.11))

$$U(n) = c_8\mathbf{P}(n\in\tau)S\left(\beta^2(1-\gamma)\sqrt{\log k}\sqrt{\frac{n/2}{9\,k}}\,\frac{\log(n/2)}{\log k}, \frac{n}{2}\right)$$
$$=: c_8\mathbf{P}(n\in\tau)s_\beta(k,n). \tag{5.26}$$

We recall also that ([12, Th. B])

$$\mathbf{P}(n\in\tau) \overset{n\to\infty}{\sim} \frac{1}{2\pi C_K\sqrt{n}}, \tag{5.27}$$

and therefore there exists $c_9 > 0$ such that for every $n\in\mathbb{N}$

$$\mathbf{P}(n\in\tau) \le \frac{c_9}{\sqrt{n}}. \tag{5.28}$$

Split the sum in (5.17) according to whether $j \le \delta k$ or not ($\delta = \delta(\eta) \in (0,1)$ is going to be chosen below). By using $S(a,L) \le 1$ (in the case $j \le \delta k$) and (5.28) we obtain

$$\sum_{j=0}^{k-1} U(j) \le c_8 + c_8c_9\sum_{j=1}^{\delta k}\frac{1}{\sqrt{j}} + c_8c_9\sum_{j=\delta k+1}^{k-1}\frac{1}{\sqrt{j}}\,s_\beta(k,j). \tag{5.29}$$

Since if c_7 is chosen sufficiently large

$$\beta^2\sqrt{\log k} \ge \sqrt{c_7 - \beta^4\log 2} \ge \sqrt{c_7}/2, \tag{5.30}$$

and since k may be made large by increasing c_7, we directly see that (5.25) implies that $s_\beta(k,j)$ may be made smaller than (say) δ for every $\delta k < j < k$ by choosing c_7 sufficiently large. Therefore (5.29) implies

$$\sum_{j=0}^{k-1} U(j) \le 4c_8c_9(\sqrt{\delta}+\delta)\sqrt{k}. \tag{5.31}$$

By choosing $\delta = \delta(\eta)$ such that $4c_8c_9(\sqrt{\delta}+\delta) \le \eta$, we have (5.17). The proof of (5.18) is absolutely analogous to the proof of (5.17) and it is therefore omitted. $\square$

5.3. **Proof of Lemma 5.4.** We introduce the notation

$$Y_L^{(i)} := \sum_{j=i+1}^{L} \frac{\delta_j}{\sqrt{j-i}}, \text{ so that } W_L = \frac{1}{\sqrt{L}\log L} \sum_{i=1}^{L-1} \delta_i Y_L^{(i)}. \tag{5.32}$$

Let us observe that, thanks to the renewal property of τ, under $\mathbf{P}(\cdot|\delta_i = 1)$, $Y_L^{(i)}$ is distributed like $Y_{L-i} := Y_{L-i}^{(0)}$ (under $\mathbf{P}$). The first step in the proof is observing that, in view of (5.28),

$$\mathbf{E}\left[\frac{1}{\sqrt{L}\log L} \sum_{i=(1-\varepsilon)L}^{L-1} \delta_i Y_L^{(i)}\right] = \frac{1}{\log L\sqrt{L}} \sum_{i=(1-\varepsilon)L}^{L-1} \sum_{j=i+1}^{L} \frac{\mathbf{P}(i\in\tau)\mathbf{P}(j-i\in\tau)}{\sqrt{j-i}} = O(\varepsilon), \tag{5.33}$$

uniformly in L, so we can focus on studying $W_{L,\varepsilon}$, defined as W_L, but stopping the sum over i at $(1-\varepsilon)L$. At this point we use that

$$\lim_{L\to\infty} \frac{Y_L}{\log L} = \frac{1}{2\pi C_K} =: \widehat{c}_K, \tag{5.34}$$

in $L^2(\mathbf{P})$ (and hence in $L^1(\mathbf{P})$). We postpone the proof of (5.34) and observe that, thanks to the properties of the logarithm, it implies that for every $\varepsilon > 0$

$$\lim_{L\to\infty} \sup_{q\in[\varepsilon,1]} \mathbf{E}\left[\left|\frac{1}{\log L}\sum_{j=1}^{qL}\frac{\delta_j}{\sqrt{j}} - \widehat{c}_K\right|\right] = 0. \tag{5.35}$$

Let us write

$$R_L := W_{L,\varepsilon} - \frac{\widehat{c}_K}{\sqrt{L}} \sum_{i=1}^{(1-\varepsilon)L} \delta_i \tag{5.36}$$

and note that $L^{-1/2}\sum_{i=1}^{(1-\varepsilon)L}\delta_i$ converges in law toward $\sqrt{(1-\varepsilon)/(2\pi C_K^2)}\,|Z|$. This follows directly by using that the event $\sum_{i=1}^{L}\delta_i \geq m$ is the event $\tau_m \leq L$ (τ_m is of course the m-th point in τ after 0) and by using the fact that τ_1 is in the domain of attraction of the positive stable law of index 1/2 [13, VI.2 and XI.5]. It suffices therefore to show that $\mathbf{E}[|R_L|]$ tends to zero. We have

$$\mathbf{E}\left[|R_L|\right] \leq \frac{1}{\sqrt{L}} \sum_{i=1}^{(1-\varepsilon)L} \mathbf{E}[\delta_i]\mathbf{E}\left[\left|\frac{Y_L^{(i)}}{\log L} - \widehat{c}_K\right| \,\middle|\, \delta_i = 1\right] = \frac{1}{\sqrt{L}} \sum_{i=1}^{(1-\varepsilon)L} \mathbf{E}[\delta_i]\mathbf{E}\left[\left|\frac{Y_{L-i}}{\log L} - \widehat{c}_K\right|\right] = o(1), \tag{5.37}$$

where in the last step we have used (5.35) and (5.28).

Note that we have also proven that $c = (2\pi)^{-3/2}C_K^{-2}$ in the statement of Lemma 5.4.

We are therefore left with the task of proving (5.34). This result has been already proven [8, Th. 6] when τ is given by the successive returns to zero of a centered, aperiodic and irreducible random walk on $\mathbb{Z}$ with bounded variance of the increment variable. Note that, by well established local limit theorems, for such a class of random walks we have

(5.27). Actually in [8] it is proven that (5.34) holds almost surely as a consequence of $\mathrm{var}_{\mathbf{P}}(Y_L) = O(\log L)$. What we are going to do is simply to re-obtain such a bound, by repeating the steps in [8] and using (5.27)-(5.28), for the general renewal processes that we consider (as a side remark: also in our generalized set-up, almost sure convergence holds).

The proof goes as follows: by using (5.27) it is straightforward to see that the limit as $L \to \infty$ of $\mathbf{E}[Y_L/\log L]$ is $\widehat{c}_K$, so that we are done if we show that $\mathrm{var}_{\mathbf{P}}(Y_L/\log L)$ vanishes as $L \to \infty$. So we start by observing that

$$\mathrm{var}_{\mathbf{P}}(Y_L) = \sum_{i,j} \frac{\mathbf{E}[\delta_i \delta_j] - \mathbf{E}[\delta_i]\mathbf{E}[\delta_j]}{\sqrt{ij}} = 2\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \frac{\mathbf{E}[\delta_i \delta_j] - \mathbf{E}[\delta_i]\mathbf{E}[\delta_j]}{\sqrt{ij}} + O(1), \quad (5.38)$$

by (5.28). Now we compute

$$\begin{aligned}
\sum_{i=1}^{L-1}\sum_{j=i+1}^{L} \frac{\mathbf{E}[\delta_i \delta_j] - \mathbf{E}[\delta_i]\mathbf{E}[\delta_j]}{\sqrt{ij}} &= \sum_{i=1}^{L-1} \frac{\mathbf{E}[\delta_i]}{\sqrt{i}} \left[\sum_{j=1}^{L-i} \frac{\mathbf{E}[\delta_j]}{\sqrt{j+i}} - \sum_{j=i+1}^{L} \frac{\mathbf{E}[\delta_j]}{\sqrt{j}} \right] \\
&\le \sum_{i=1}^{L-1} \frac{\mathbf{E}[\delta_i]}{\sqrt{i}} \left[\sum_{j=1}^{L-i} \frac{\mathbf{E}[\delta_j]}{\sqrt{j+i}} - \sum_{j=i+1}^{L} \frac{\mathbf{E}[\delta_j]}{\sqrt{j+i}} \right] \\
&\le \sum_{i=1}^{L-1} \frac{\mathbf{E}[\delta_i]}{\sqrt{i}} \sum_{j=1}^{i} \frac{\mathbf{E}[\delta_j]}{\sqrt{j+i}} \\
\le \sum_{i=1}^{L-1} \frac{\mathbf{E}[\delta_i]}{i} \sum_{j=1}^{i} \mathbf{E}[\delta_j] &\le c_9^2 \sum_{i=1}^{L-1} \frac{1}{i^{3/2}} \sum_{j=1}^{i} \frac{1}{j^{1/2}} = O(\log L),
\end{aligned} \quad (5.39)$$

where, in the last line, we have used (5.28). In view of (5.38), we have obtained $\mathrm{var}_{\mathbf{P}}(Y_L) = O(\log L)$ and the proof (5.34), and therefore of Lemma 5.4 is complete. □

Appendix A. Some technical results and useful estimates

A.1. **Two results on renewal processes.** The first result concerns the non-disordered pinning model and is well known:

Lemma A.1. *Let $K(\cdot)$ be a probability on $\mathbb{N}$ which satisfies* (3.1) *for some $\alpha > 0$. If $h < 0$, we have that*

$$\sum_{\ell=1}^{N} \sum_{i_0:=0<i_1<\ldots<i_\ell=N} e^{h\ell} \prod_{r=1}^{\ell} K(i_r - i_{r-1}) \stackrel{N\to\infty}{\longrightarrow} 0. \quad (A.1)$$

This is implied by [18, Th. 2.2], since the left-hand side of (A.1) is nothing but the partition function of the homogeneous pinning model of length N, whose critical point is $h_c = 0$ (*cf.* also (3.4)).

The second fact we need is

Lemma A.2. *There exists a positive constant c, which depends only on $K(\cdot)$, such that for every positive function $f_N(\tau)$ which depends only on $\tau \cap \{1, \ldots, N\}$ one has*

$$\sup_{N>0} \frac{\mathbf{E}[f_N(\tau) | 2N \in \tau]}{\mathbf{E}[f_N(\tau)]} \le c. \quad (A.2)$$

Proof. The statement follows by writing $f_N(\tau)$ as $f_N(\tau)\sum_{n=0}^N \mathbf{1}_{\{X_N=n\}}$, where X_N is the last renewal epoch up to (and including) N, and using the bound

$$\sup_N \max_{n=0,\dots,N} \frac{\mathbf{P}(X_N=n|2N\in\tau)}{\mathbf{P}(X_N=n)} =: c < \infty,$$

which is equation (A.15) in [10] (this has been proven also in [33], where the proof is repeated to show that c can be chosen as a function of α only). ☐

A.2. **Proof of** (5.15)**.** Defining the event

$$\Omega_{\underline{n},\underline{j}} := \{N\in\tau \text{ and } \{j_{r-1},\dots,n_r\}\cap\tau = \{j_{r-1},n_r\} \text{ for all } r=1,\dots,\ell\}, \tag{A.3}$$

with the convention that $j_0 := 0$, we have

$$\widehat{Z}_\omega^{(i_1,\dots,i_\ell)} = \sum_{n_1\in B_{i_1}} \cdots \sum_{\substack{n_\ell\in B_{N/k}:\\ n_\ell\ge n_{\ell-1}+k}} \mathbf{E}\left[e^{\sum_{n=1}^N(\beta\omega_n+h-\beta^2/2)\delta_n};\Omega_{\underline{n},\underline{j}}\right]. \tag{A.4}$$

Since $\widetilde{\mathbb{P}}$ is a Gaussian measure and $\delta_i^2=\delta_i$ for every i, the computation of $\widetilde{\mathbb{E}}\widehat{Z}_\omega^{(i_1,\dots,i_\ell)}$ is immediate:

$$\widetilde{\mathbb{E}}\widehat{Z}_\omega^{(i_1,\dots,i_\ell)} = \sum_{n_1\in B_{i_1}} \cdots \sum_{\substack{n_\ell\in B_{N/k}:\\ n_\ell\ge n_{\ell-1}+k}} \mathbf{E}\left[e^{h\sum_{n=1}^N\delta_n - \beta^2/2\sum_{i,j=1}^N \mathcal{C}_{ij}\delta_i\delta_j};\Omega_{\underline{n},\underline{j}}\right]. \tag{A.5}$$

In view of $\mathcal{C}_{ij}\ge 0$, we obtain an upper bound by neglecting in the exponent the terms such that $n_r\le i\le j_r$ and $n_{r'}\le j\le j_{r'}$ with $r\ne r'$. At that point, the $\mathbf{E}$ average may be factorized, by using the renewal property, and we obtain (recall that $\mathcal{C}_{ii}=0$)

$$\begin{aligned}\widetilde{\mathbb{E}}\widehat{Z}_\omega^{(i_1,\dots,i_\ell)} \le &\sum_{n_1\in B_{i_1}} \cdots \sum_{\substack{n_\ell\in B_{N/k}:\\ n_\ell\ge n_{\ell-1}+k}} K(n_1)\dots K(n_\ell-j_{\ell-1})\\ &\times\prod_{r=1}^{\ell}\mathbf{E}\left[e^{h\sum_{i=n_r}^{j_r}\delta_i-\beta^2\sum_{n_r\le i<j\le j_r}\mathcal{C}_{ij}\delta_i\delta_j}\mathbf{1}_{\{j_r\in\tau\}}\middle| n_r\in\tau\right],\end{aligned} \tag{A.6}$$

with the convention that $j_\ell := N$. We are left with the task of proving that

$$\mathbf{E}\left[e^{h\sum_{i=n_r}^{j_r}\delta_i-\beta^2\sum_{n_r\le i<j\le j_r}\mathcal{C}_{ij}\delta_i\delta_j}\mathbf{1}_{\{j_r\in\tau\}}\middle| n_r\in\tau\right] \le U(j_r-n_r), \tag{A.7}$$

with $U(\cdot)$ satisfying (5.16). We remark first of all that the left-hand side of (A.7) equals

$$\mathbf{P}(j_r-n_r\in\tau)\mathbf{E}\left[e^{h\sum_{i=n_r}^{j_r}\delta_i-\beta^2\sum_{n_r\le i<j\le j_r}\mathcal{C}_{ij}\delta_i\delta_j}\middle| n_r\in\tau, j_r\in\tau\right]. \tag{A.8}$$

Since by construction $j_r-n_r<k(h)=\lfloor 1/h\rfloor$, one has

$$e^{h\sum_{i=n_r}^{j_r}\delta_i} \le e. \tag{A.9}$$

As for the remaining average, assume without loss of generality that $|\{n_r,n_r+1,\dots,j_r\}\cap B_{i_r}|\ge (j_r-n_r)/2$ (if this is not the case, the inequality clearly holds with B_{i_r} replaced

by B_{i_r+1} and the arguments which follow are trivially modified). Then,

$$\mathbf{E}\left[e^{-\beta^2 \sum_{n_r \le i<j\le j_r} \mathcal{C}_{ij}\delta_i\delta_j} \middle| n_r \in \tau, j_r \in \tau\right] \le \mathbf{E}\left[\exp\left(-\beta^2 \sum_{0<i<j\le (j_r-n_r)/2} \delta_i\delta_j H_{ij}\right)\middle| j_r - n_r \in \tau\right]. \quad \text{(A.10)}$$

Finally, the conditioning in (A.10) can be eliminated using Lemma A.2, and (5.15) is proved. □

A.3. **Proof of Lemma 5.2.** In this proof (and in the statement) two positive numbers C_1 and C_2 appear. C_1 is going to change along with the steps of the proof: it depends on η, k and on $K(\cdot)$. C_2 instead is chosen once and for all below and it depends only on $K(\cdot)$. We start by giving a name to the right-hand side of (5.15):

$$Q := \sum_{n_1\in B_{i_1}} \sum_{j_1=n_1}^{n_1+k-1} \sum_{\substack{n_2\in B_{i_2}:\\ n_2\ge n_1+k}} \sum_{j_2=n_2}^{n_2+k-1} \cdots \sum_{\substack{n_{\ell-1}\in B_{i_{\ell-1}}:\\ n_{\ell-1}\ge n_{\ell-2}+k}} \sum_{j_{\ell-1}=n_{\ell-1}}^{n_{\ell-1}+k-1} \sum_{\substack{n_\ell\in B_{N/k}:\\ n_\ell\ge n_{\ell-1}+k}} K(n_1)\ldots K(n_\ell-j_{\ell-1})U(j_1-n_1)\ldots U(j_{\ell-1}-n_{\ell-1})U(N-n_\ell). \quad \text{(A.11)}$$

Since $N-n_\ell<k$, we can get rid of $U(N-n_\ell)$ $(\le c_8\mathbf{P}(N-n_\ell\in\tau))$ and of the right-most sum (on n_ℓ), replacing n_ℓ by N, by paying a price that depends on k and $K(\cdot)$ (this price goes into C_1). Therefore we have

$$Q \le C_1 \sum_{n_1\in B_{i_1}} \cdots \sum_{j_{\ell-1}=n_{\ell-1}}^{n_{\ell-1}+k-1} K(n_1)\ldots K(n_\ell-j_{\ell-1})U(j_1-n_1)\ldots U(j_{\ell-1}-n_{\ell-1}), \quad \text{(A.12)}$$

where by convention from now on $n_\ell := N$. Now we single out the long jumps. The set of long jump arrival points is defined as

$$J = J(i_1,i_2,\ldots,i_\ell) := \{r:\, 1\le r\le \ell,\, i_r > i_{r-1}+2\}, \quad \text{(A.13)}$$

and the definition guarantees that a long jump $\{j_{r-1},\ldots,n_r\}$ contains at least one whole block with no renewal point inside. For $r\in J$ we use the bound

$$K(n_r-j_{r-1}) \le \frac{C_2}{(i_r-i_{r-1})^{3/2}k^{3/2}}, \quad \text{(A.14)}$$

and we stress that we may and do choose C_2 depending only on $K(\cdot)$. For later use, we choose $C_2\ge 2^{3/2}$. This leads to

$$Q \le C_1\, k^{-3|J|/2} \prod_{r\in J}\frac{C_2}{(i_r-i_{r-1})^{3/2}} \times \sum_{n_1\in B_{i_1}} \cdots \sum_{j_{\ell-1}=n_{\ell-1}}^{n_{\ell-1}+k-1} \left(\prod_{r\in\{1,\ldots,\ell\}\setminus J} K(n_r-j_{r-1})\right) U(j_1-n_1)\ldots U(j_{\ell-1}-n_{\ell-1}). \quad \text{(A.15)}$$

Now we perform the sums in (A.15) and bound the outcome by using the assumptions (5.17) and (5.18).

We first sum over j_{r-1}, $r\in J$, keeping of course into account the constraint $0\le j_{r-1}-n_{r-1}<k$. By using (5.17) such sum yields at most $(\eta\sqrt{k})^{|J|}$ if $1\notin J$. If $1\in J$, for

$r = 1$ then $j_0 = 0$ and there is no summation: we can still bound the sum by $(\eta\sqrt{k})^{|J|}$, provided that we change the constant C_1.

Second, we sum over j_{r-1}, n_r for $r \in \{1, \ldots, \ell\} \setminus J$ and use (5.18). Once again we have to treat separately the case $r = 1$, as above. But if $1 \notin \{1, \ldots, \ell\} \setminus J$ we directly see that the summation is bounded by $\eta^{\ell - |J|}$.

Finally, we have to sum over n_r, for $r \in J$. The summand does not depend on these variables anymore, so this gives at most $k^{|J|}$.

Putting these estimates together we obtain

$$Q \le C_1 \frac{(\eta\sqrt{k})^{|J|}\eta^{\ell-|J|}k^{|J|}}{k^{3|J|/2}} \prod_{r \in J} \frac{C_2}{(i_r - i_{r-1})^{3/2}} \le C_1 \eta^\ell C_2^\ell \prod_{r=1}^{\ell} \frac{1}{(i_r - i_{r-1})^{3/2}}, \tag{A.16}$$

where, in the last step, we have used $C_2 \ge 2^{3/2}$. The proof of Lemma 5.2 is therefore complete. □

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